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Differential Constants of Motion for Systems of Free Gravitating Particles. II. General Relativity

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Differential constants of motion for systems of freely gravitating particles in general relativity are first defined and then determined. It is shown that they are all consequences of the local simultaneity conservation property of general relativity. It is proved, further, that the restriction to vacuum conditions does not change the set of differential constants of motion, excluding the nonphysical cases of space-time of dimension 2 or 3. Another consequence is that nothing can be inferred from local (in space and time) measurements about the orientation of a laboratory in free fall relative to Fermi transported axes. A similar property exists in Newton's theory.

1. INTRODUCTION

The problem of differential constants of motion (DCMs) for a continuum consisting of freely gravitating, noncolliding particles was studied in the preceding paper (Enosh and Kovetz, 1977) (hereafter referred to as Paper I) in the framework of Newton's theory. Here we study the analogous problem in Einstein's theory. Since many of the ideas, methods, notations, and conventions of this paper can be found in Paper I, sometimes in more detail, it would be advisable to have read Paper I first. However, we believe that (apart from the explicitly noted references) the present paper is self-contained.

Along the history of any particle surrounded by others, all in free fall and all carrying clocks, we can speak of differential quantities. Generally speaking these are arbitrary differentiable functions of some arguments which form a finite subset of a certain infinite set. The latter is determined in the following

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way: Certain functions describe space-time, the particles' motion and the (clocks') proper time. The values of all the derivatives of these functions are the elements of the infinite set mentioned. In order to derive meaningful results we shall consider in particular the covariant differential quantities, that is, the quantities that are unambiguously determined by the physical–mathematical structure of the system. Technically, the covariant quantities are the quantities that are independent of transformations of space-time coordinates. *The DCMs are those differential quantities that remain constant along the history of every particle in every continuum in every gravitational field.* Obviously, the DCMs must be covariant; indeed it follows below.

Throughout this paper lowercase Latin and Greek and capital Latin indices run over the ranges $\{0, 1, 2, 3\}$, $\{1, 2, 3\}$, and $\{\bar{1}, \bar{2}, \ldots, \bar{6}\}$, respectively (exceptions are noted explicitly). The matrix tensor g_{ij} has the signature of $\eta_{ij} \equiv \mathrm{diag}\,(+1, -1, -1, -1)$ and $\{^{i}_{jk}\}$ are the related Christoffel symbols. Partial derivatives with respect to a parameter distinguished by an index are sometimes denoted by a diagonal stroke followed by the index (e.g., $t_{A/B} \equiv \partial t_A/\partial d^B$, $g_{ij/k} \equiv \partial g_{ij}/\partial x^k$). Covariant derivatives with respect to a parameter are denoted by means of δ/δ (e.g., $\delta U^i/\delta s$), and with respect to coordinates by means of a semicolon (e.g., $g_{ij;k} \equiv \delta g_{ij}/\delta x^k$). Parentheses and square brackets around indices denote the symmetric and the antisymmetric part respectively. Riemann's tensor is chosen so that $2\xi^i{}_{;[j;k]} = R^i{}_{ajk}\xi^a$. For scalar products between 4-vectors we sometimes write $(AB) \equiv A_i B^i$. The general summation convention is strictly kept: A letter occurring twice, no matter where, as an index in a product should be automatically summed over the whole range of the index. As usual we mark the important equations by a running number. In addition, however, we introduce in some sections a notation by letters for equations of local importance.

In order to find the DCMs one needs to solve systems of homogeneous linear partial differential equations of the first order for a single unknown function; We shall apply to them the technique of the crossing process outlined, for example, in Schouten (1954). To describe our operations economically we introduce the following notation: Let $F(y)$ satisfy

$$a^i \frac{\partial}{\partial y^i} F = 0 \tag{a}$$

$$b^i \frac{\partial}{\partial y^i} F = 0 \tag{b}$$

(Here and in the following the indices $i, j, \ldots$ run over any finite set.) Then F also satisfies equation (c)

$$a^j \frac{\partial}{\partial y^j} \left(b^i \frac{\partial}{\partial y^i} F\right) - b^j \frac{\partial}{\partial y^j} \left(a^i \frac{\partial}{\partial y^i} F\right) = 0 \tag{c}$$

obtained, we say, by "crossing of (a) and (b)." Equation (c) is again a homogeneous linear differential equation of the first order:

$$\text{(c)} \qquad c^i \frac{\partial}{\partial y^i} F = 0, \qquad c^i \equiv a^j b^i{}_{|j} - b^j a^i{}_{|j}$$

We shall write symbolically $[a, b] = (c)$.

2. DIFFERENTIAL QUANTITIES AND THE DEFINITION OF A DIFFERENTIAL CONSTANT OF MOTION (DCM)

2.1. The Mathematical Structure of the System. Let x^i be arbitrary coordinates in space-time. Six parameters, $(d) \equiv (d^A)$, serve to identify all the possible motions of free particles. The functions that describe these motions and the proper time, s, along them are given by $x^i = \bar{x}^i(s; d)$, $[\bar{x} = \bar{x}(s; d)]$. In addition to these, the metric's functions, $g_{ij}(x)$, are also available. The differential quantities (along the history of a certain particle d) are constructed out of s and the derivatives of $\bar{x}(s; d)$ and $g_{ij}(\bar{x})$.

The functions $\bar{x}(s; d)$ and $g_{ij}(x)$ satisfy

$$\frac{\partial^2 \bar{x}^i}{\partial s^2} + \begin{Bmatrix} i \\ j \quad k \end{Bmatrix} \frac{\partial \bar{x}^j}{\partial s} \frac{\partial \bar{x}^k}{\partial s} = 0, \qquad g_{ij} \frac{\partial \bar{x}^i}{\partial s} \frac{\partial \bar{x}^j}{\partial s} = 1 \tag{2.1}$$

also, $\bar{x}(s; d)$ should include all the possible free motions. We give different form to these restrictions as follows. Let us define the following 4-vectors at $\bar{x}(s; d)$:

$$U^i \equiv \frac{\partial \bar{x}^i}{\partial s}, \qquad D_A{}^i \equiv \frac{\partial \bar{x}^i}{\partial d^A}, \qquad D^i_{A_1 \cdots A_n} \equiv \frac{\delta}{\delta d^{A_n}} \cdots \frac{\delta}{\delta d^{A_2}} D^i_{A_1},$$

$$U^i_{A_1 \cdots A_n} \equiv \frac{\delta}{\delta d^{A_n}} \cdots \frac{\delta}{\delta d^{A_1}} U^i \tag{2.2}$$

They satisfy

$$\frac{\delta}{\delta s} D_A{}^i = U_A{}^i, \qquad D^i_{AB} = D^i_{BA} \tag{2.3}$$

$$\frac{\delta}{\delta s} U^i = 0, \qquad (UU) = 1 \tag{2.4}$$

Equations (2.4) are equivalent to equations (2.1). Let us denote

$$(U^i) = (U^0, U^1, U^2, U^3), \qquad (D_A{}^i) = (D_A{}^0, D_A{}^1, D_A{}^2, D_A{}^3),$$
$$(U_A{}^i) = (U_A{}^0, U_A{}^1, U_A{}^2, U_A{}^3)$$

Then, a necessary and sufficient condition for $\bar{x}(s; d)$ to include (locally) all the geodesics is

$$\det \begin{pmatrix} (U^i) & 0 \\ (D_{\bar{1}}^{\,i}) & (U_{\bar{1}}^{\,i}) \\ \vdots & \vdots \\ (D_{\bar{6}}^{\,i}) & (U_{\bar{6}}^{\,i}) \\ 0 & (U^i) \end{pmatrix} \neq 0 \tag{2.5}$$

By means of a construction it is possible to prove the following statement. *Given a metric in space-time, certain* $(s_0; d_0)$, $\bar{x}(s_0; d_0)$ *and a system of coordinates in a neighborhood of* $\bar{x}(s_0; d_0)$, *then apart from the symmetry in the indices* $A_1, \ldots, A_k$ *and the equations obtained by differentiating the equation* $(UU) = 1$ *with respect to the* d^A, *all the quantities* $\{U^i, U^i_{(A_1 \cdots A_k)}, D^i_{(A_1 \cdots A_k)}\}_{k=1}^{\infty}$ *at* $(s_0; d_0)$ *are functionally independent.* [They have to satisfy (2.5), too; but the inequality does not reduce the set of functionally independent quantities.] Also, *those quantities among them that are restricted by* $k \leqslant K$ *determine (and are determined) by* $\{U^i, U^i_{A_1 \cdots A_k}, D^i_{A_1 \cdots A_k}\}_{k=1}^{K}$ *at* $(s_0; d_0)$.

2.2. The Generalized Covariant Differential Quantities. In order to treat the differential quantities along a certain geodesic, d, we attach an auxiliary triad of spatial axes to it, which form together with U^i an orthonormal tetrad along it. This particular choice enriches the given structure of the system and, therefore, it enlarges the set of covariant differential quantities to what we call the set of generalized covariant differential quantities. To be definite, a generalized covariant differential quantity is an arbitrary function of the "ordinary" arguments, s and the derivatives of $\bar{x}(s; d)$ and $g_{ij}(x)$, and of the extra arguments, the chosen triad components, that is independent of space-time coordinates transformations, provided the triad components also transform (as four vector components). In principle this choice may enlarge the set of DCMs (defined later in Section 2.4) too. We hope, however, that we shall be able to find the "ordinary" covariant DCMs in the larger set. So, we concentrate on the generalized covariant quantities in the following. In order to gain some advantage out of this approach we choose the spatial axes to obey the parallel transport law, which has a physical meaning (Synge, 1960). Thus, for example, if a DCM that depends on the rotation of the axes does exist, then it is possible to learn something about the orientation of a laboratory (perhaps to fix it completely) relative to parallel-transported axes from local measurements only. (Then one can determine the laboratory orientation after a finite time interval without keeping parallel-transported axes all the time.)

Now, at a given event $\bar{x}(s; d)$ on the geodesic d we choose an arbitrary system of coordinates provided its origin and its axes at the origin coincide with $\bar{x}(s; d)$ and the given tetrad at $\bar{x}(s; d)$, respectively. We consider functions of s and of the derivatives of g_{ij} and $\bar{x}^i(s; d)$ in these coordinates at the origin. The geodesic equation (2.1) enables one to express derivatives of $\bar{x}(s; d)$ with at least two derivations with respect to s in terms of lower derivatives of $\bar{x}(s; d)$ and derivatives of g_{ij}. At the origin $g_{ij} = \eta_{ij}$ and $\partial\bar{x}^i/\partial s = \delta_0{}^i$; therefore it is sufficient to consider the quantities

$$s, \frac{\partial \bar{x}^i}{\partial d^A}, \frac{\partial^2 \bar{x}^i}{\partial d^A \partial d^B}, \frac{\partial^3 \bar{x}^i}{\partial d^A \partial d^B \partial d^C}, \ldots, \frac{\partial^2 \bar{x}^i}{\partial s \partial d^A}, \frac{\partial^3 \bar{x}^i}{\partial s \partial d^A \partial d^B}, \ldots, g_{ij/k}, g_{ij/k/l}, \ldots$$

The determination of these is completely equivalent to the determination of

$$s, D_A{}^i, D^i_{(AB)}, D^i_{(ABC)}, \ldots, U_A{}^i, U^i_{(AB)}, \ldots, g_{ij/k}, g_{ij/k/l}, \ldots$$

as follows by induction. Among these quantities only the derivatives of g_{ij} still depend on the freedom left in the choice of the coordinates. However, every generalized covariant differential quantity has to be a certain function of the previous quantities in *every* system of coordinates, the axes of which at the origin coincide with the given tetrad. We may choose, in particular, the normal geodetic system of coordinates (Schouten, 1954, p. 155), which is determined completely by the given tetrad. The derivatives of g_{ij} in this system of coordinates are generalized covariant differential quantities. Since, further, these quantities are completely determined by R_{ijkl} and its covariant derivatives, the

$$s, D_A{}^i, D^i_{(AB)}, D^i_{(ABC)}, \ldots, U_A{}^i, U^i_{(AB)}, \ldots, R_{ijkl}, R_{ijkl;m}, R_{ijkl;m;n}, \ldots \quad (2.6)$$

form a complete set for the construction of generalized covariant differential quantities. The quantities (2.6) should be understood as the components of the relevant tensors on the given orthonormal tetrad; as such they are scalars. *From now on all the components of tensors should be understood as taken with respect to the given tetrad.*

We wish to find a functionally independent complete set (a basis) for the generalized covariant differential quantities. To this end we make use of the quantities $S_{i_1 i_2 \cdots i_n}$ $(n = 4, 5, \ldots)$,

$$S_{ijkl} \equiv \tfrac{1}{3}(R_{ikjl} + R_{jkil}), \qquad S_{ijklmn\ldots} \equiv S_{ij(kl;m;n\ldots)} \quad (2.7)$$

and of their properties which can be found in the Appendix. These quantities were introduced by Penrose (1960); his definition equals (2.7) up to a factor only. Then, taking also into consideration the statement following equation (2.5) (by which the $U^0_{(A_1 \cdots A_n)}$ are omitted), we find that, *apart from the symmetries in the indices $A, B, C, \ldots$, and the symmetries*

$$S_{ijklmn\ldots} = S_{(ij)(klmn\ldots)}, \qquad S_{i(jklm\ldots)} = 0 \quad (2.8)$$

the quantities (scalars!)

$$s, D_A{}^i, D^i_{(AB)}, D^i_{(ABC)}, \ldots, U_A{}^\alpha, U^\alpha_{(AB)}, \ldots, S_{ijkl}, S_{ijklm}, \ldots \quad (2.9)$$

are functionally independent and form a basis for the generalized covariant differential quantities.

The independence of the quantities (2.9) enables us to classify them as follows. A quantity from (2.9) is of order n, or of A-order n, if it has n indices of type $A, B, \ldots$. We generalize this to any generalized covariant differential quantity in the natural way, according to the highest order of its nontrivial variables in its representation as a function of the quantities (2.9). We introduce, in addition, the notation $[An]$, which may be accompanied with indices if necessary, to represent a *polynomial* in the quantities (2.9) of A-order n *at most*. For example: $U^0_{A_1\cdots A_n} = [A(n-1)]_{A_1\cdots A_n}$, [as implied by (2.2) and (2.4)]; $U^i_{A_1\cdots A_n} = [An]_{A_1\cdots A_n}$; $D^i_{A_1\cdots A_n} = [An]_{A_1\cdots A_n}$.

Equation (2.5) imposes a further restriction on the quantities (2.9); but as an inequality it does not reduce further the set of functionally independent quantities among them. Since now $U^i = \delta_0{}^i$, (2.5) takes the form

$$\det\begin{pmatrix} (D_{\bar{1}}{}^{\alpha}) & (U_{\bar{1}}{}^{\alpha}) \\ \vdots & \vdots \\ (D_{\bar{6}}{}^{\alpha}) & (U_{\bar{6}}{}^{\alpha}) \end{pmatrix} \neq 0 \tag{2.10}$$

where

$$(U_A{}^{\alpha}) = (U_A{}^1, U_A{}^2, U_A{}^3), \qquad (D_A{}^{\alpha}) = (D_A{}^1, D_A{}^2, D_A{}^3)$$

We shall make use of (2.10) later

2.3. Modification of the Basis (2.9). We shall choose another basis for the generalized covariant differential quantities as follows. Given any U^i, $D_A{}^i$, $U_A{}^i$ that satisfy (2.5), it is easy to show that their covariant components satisfy

$$\det\begin{pmatrix} -(U_i) & 0 \\ -(D_{\bar{1}i}) & (U_{\bar{1}i}) \\ \vdots & \vdots \\ -(D_{\bar{6}i}) & (U_{\bar{6}i}) \\ 0 & (U_i) \end{pmatrix} \neq 0$$

where

$$(U_i) = (U_0, U_1, U_2, U_3), \qquad (D_{Ai}) = (D_{A0}, D_{A1}, D_{A2}, D_{A3}),$$
$$(U_{Ai}) = (U_{A0}, U_{A1}, U_{A2}, U_{A3})$$

Therefore, the rows of the previous matrix form a linear basis in the space of 8-tuples, and, in particular, we may represent every 8-tuple by its eight 8-Cartesian scalar products with these basis elements. Assume that for a given $n (n = 2, 3, \ldots)$ the quantities up to order $n-1$ are given. Now, for given $A_1, \ldots, A_n$, fixing of $\langle D^i_{(A_1\cdots A_n)}, U^{\alpha}_{(A_1\cdots A_n)}\rangle$ is equivalent to fixing of $\langle D^i_{(A_1\cdots A_n)}, U^i_{(A_1\cdots A_n)}\rangle$ since $U^0_{(A_1\cdots A_n)} = [A(n-1)]_{A_1\cdots A_n}$, and this is equivalent to fixing of the above-mentioned eight scalar products, namely (up to a sign), $\langle (UU_{(A_1\cdots A_n)}), h_{AA_1\cdots A_n}, k_{A_1\cdots A_n}\rangle$, where

$$h_{A_1\cdots A_n} \equiv -(D_{A_1}U_{(A_2\cdots A_n)}) + (U_{A_1}D_{(A_2\cdots A_n)}) \qquad (n = 2, 3, \ldots) \quad (2.11)$$

$$k_{A_1\cdots A_n} \equiv (UD_{(A_1\cdots A_n)}) = D^0_{(A_1\cdots A_n)} \qquad (n = 2, 3, \ldots) \qquad (2.12)$$

and this is equivalent to fixing of $\langle h_{AA_1\cdots A_n}, k_{A_1\cdots A_n}\rangle$, since $(UU_{(A_1\cdots A_n)}) = U^0_{(A_1\cdots A_n)}$ is known. Therefore, by an inductive process we may replace $\langle U^\alpha_{(A_1\cdots A_n)}, D^i_{(A_1\cdots A_n)}\rangle$ $(n = 2, 3, \ldots)$ in (2.9) by $\langle h_{AA_1\cdots A_n}, k_{A_1\cdots A_n}\rangle$. We obtain another basis

$$s,\ D_A{}^i,\ U_A{}^\alpha,\ k_{AB},\ h_{ABC},\ k_{ABC},\ h_{ABCD}, \ldots,\ S_{ijkl},\ S_{ijklm}, \ldots \qquad (2.13)$$

in which the quantities are functionally independent, apart from the symmetries (2.8) and

$$h_{A_1A_2\cdots A_n} = h_{A_1(A_2\cdots A_n)} \qquad (n = 3, 4, \ldots) \qquad (2.14)$$

$$k_{A_1\cdots A_n} = k_{(A_1\cdots A_n)} \qquad (n = 2, 3, \ldots) \qquad (2.15)$$

Also, the quantities

$$\{s,\ D_A{}^i,\ U_A{}^\alpha,\ S_{ijkl},\ S_{ijklm}, \ldots\} U \{k_{A_1\cdots A_n}\}_{n=2}^N U \{h_{AA_1\cdots A_n}\}_{n=2}^N$$

form a basis for all the generalized covariant quantities of order N.

We now perform some more modifications of the basis (2.13). By Lemma 1 in the Appendix of Paper I and by (2.14) we may replace every $h_{A_1\cdots A_n}$ $(n = 3, 4, \ldots)$ in (2.13) by $h_{(A_1\cdots A_n)}$ and $h_{[A_1A_2]A_3\cdots A_n}$ which are arbitrary apart from symmetry of $h_{(A_1\cdots A_n)}$ and

$$h_{[A_1A_2]A_3\cdots A_n} = h_{[A_1A_2](A_3\cdots A_n)}, \qquad h_{[[A_1A_2]A_3]A_4\cdots A_n} = 0 \qquad (n = 3, 4, \ldots) \quad (2.16)$$

Then, by an inductive process, we may add to each quantity of (2.13) a term (of the same symmetry) of lower order. Firstly, let us replace every $h_{(A_1\cdots A_n)}$ by $H_{A_1\cdots A_n}$, where

$$H_{A_1\cdots A_n} \equiv h_{(A_1A_2A_3/A_4/\cdots/A_n)} \qquad (n = 3, 4, \ldots) \qquad (2.17)$$

Indeed, $H_{A_1\cdots A_n}$ is symmetric as $h_{(A_1\cdots A_n)}$ is, and they differ from each other by a term of order $n - 2$ at most [as implied by (2.17) and (2.11) by induction]. Now we turn to modify $k_{A_1\cdots A_{n-1}}$, $h_{[A_1A_2]A_3\cdots A_n}$. To this end we define the differential quantities

$$t_A \equiv -(UD_A), \qquad t_{A_1\cdots A_n} \equiv t_{A_1/A_2/\cdots A_n} \qquad (2.18)$$

which obviously satisfy

$$t_{A_1A_2\cdots A_n} = t_{A_1(A_2\cdots A_n)} \qquad (n = 2, 3, \ldots) \qquad (2.19)$$

Equation (2.18) implies

$$t_{A_1A_2} = -(UD_{A_1A_2}) - (U_{A_2}D_{A_1})$$

Hence, with the aid of (2.3)

$$t_{[A_1A_2]} = \tfrac{1}{2}[(D_{A_2}U_{A_1}) - (D_{A_1}U_{A_2})] \tag{2.20}$$

and by induction

$$t_{[A_1A_2]A_3\cdots A_n} = -(D_{[A_1}U_{A_2]A_3\cdots A_n}) + (U_{[A_1}D_{A_2]A_3\cdots A_n}) + [A(n-2)]_{A_1\cdots A_n} \qquad (n = 3, 4, \ldots)$$

However, since

$$D^i_{A_2\cdots A_n} = D^i_{(A_2\cdots A_n)} + [A(n-2)]_{A_2\cdots A_n};$$
$$U^i_{A_2\cdots A_n} = U^i_{(A_2\cdots A_n)} + [A(n-2)]_{A_2\cdots A_n}$$

it follows with the aid of (2.11) that

$$t_{[A_1A_2]A_3\cdots A_n} = h_{[A_1A_2]A_3\cdots A_n} + [A(n-2)]_{A_1\cdots A_n} \qquad (n = 3, 4, \ldots) \tag{2.21}$$

Another immediate consequence of (2.18) is

$$t_{A_1\cdots A_n} = -(UD_{A_1\cdots A_n}) + [A(n-1)]_{A_1\cdots A_n}$$

This equation and (2.12) imply

$$t_{(A_1\cdots A_n)} = k_{A_1\cdots A_n} + [A(n-1)]_{A_1\cdots A_n} \qquad (n = 2, 3, \ldots) \tag{2.22}$$

Since $t_{[A_1A_2]A_3\cdots A_n}$ and $t_{(A_1\cdots A_n)}$ have the same symmetries of $h_{[A_1A_2]A_3\cdots A_n}$, (2.16), and of $k_{A_1\cdots A_n}$, (2.15), respectively [a consequence of (2.19) and of Lemma 1 in the Appendix of Paper I], it follows from (2.21) and (2.22) that we may replace every $k_{A_1\cdots A_n}$ $(n = 2, 3, \ldots)$, and every $h_{[A_1A_2]A_3\cdots A_n}$ $(n = 3, 4, \ldots)$ in the basis (2.13) by $t_{(A_1\cdots A_n)}$ and $t_{[A_1A_2]A_3\cdots A_n}$, respectively. Thus, starting at (2.13) we obtain, apart from the symmetries,

$$H_{A_1\cdots A_n} = H_{(A_1\cdots A_n)} \qquad (n = 3, 4, \ldots) \tag{2.23}$$

$$t_{[A_1A_2]A_3\cdots A_n} = t_{[A_1A_2](A_3\cdots A_n)} \qquad (n = 3, 4, \ldots) \tag{2.24}$$

and the symmetry of $t_{(A_1\cdots A_n)}$, the quantities

$$s,\ D_A{}^i,\ U_A{}^\alpha,\ t_{(AB)},\ H_{ABC},\ t_{[AB]C},\ t_{(ABC)},\ H_{ABCD},\ t_{[AB]CD},\ \ldots,\ S_{ijkl},\ S_{ijklm},\ \ldots \tag{2.25}$$

are functionally independent and form a basis for the generalized covariant differential quantities. Now we apply Lemma 1 in the Appendix of Paper I again to $t_{(A_1\cdots A_n)}$ and $t_{[A_1A_2]A_3\cdots A_n}$ and find [with the aid of (2.24) and (2.19)] the following result:

Apart from the symmetries (2.8), (2.19), (2.23) and that of $t_{(AB)}$ the quantities

$$s,\ D_A{}^i,\ U_A{}^\alpha,\ t_{(AB)},\ H_{ABC},\ t_{ABC},\ H_{ABCD},\ t_{ABCD},\ \ldots,\ S_{ijkl},\ S_{ijklm},\ \ldots \tag{2.26}$$

are functionally independent and form a basis for the generalized covariant *differential quantities.* [Equations (2.7), (2.11), (2.17), and (2.18) define the $S_{ijklm\ldots}$, $H_{ABC\ldots}$, $t_{ABC\ldots}$.] We shall not modify the basis (2.26) any further.

Our operations imply that the $\{s, D_A{}^i, U_A{}^\alpha, S_{ijkl}, S_{ijklm}, \ldots\}$ form a basis for the first-order quantities, the $\{s, D_A{}^i, U_A{}^\alpha, t_{(AB)}, H_{ABC}, t_{[AB]C}, S_{ijkl}, \ldots\}$ form a basis for the second-order quantities, and the

$$\{s, D_A{}^i, U_A{}^\alpha, t_{(A_1A_2)}, H_{A_1A_2A_3}, t_{A_1A_2A_3}, \ldots, H_{A_1A_2\cdots A_n}, t_{A_1\cdots A_n},$$
$$H_{A_1\cdots A_{n+1}}, t_{[A_1A_2]A_3\cdots A_{n+1}}, S_{ijkl}, S_{ijklm}, \ldots\}$$

form a basis for the quantities of order n ($n > 2$).

We introduce a further classification of the generalized covariant differential quantities: Such a quantity is of S-order n if as a function of the quantities of (2.26) the $S_{abi_1\cdots i_n}$ occur among its nontrivial arguments while the $\{S_{abi_1\cdots i_k}\}_{k\geqslant n+1}$ do not. Also the notation $[An, Sm]$ (sometimes accompanied by indices) will stand for a polynomial in the variables (2.26) of A-order n and S-order m, *at most*; and $[Sn]$ will stand for a polynomial in the $\{S_{abi_1\cdots i_k}\}_{k=2}^{n}$ only [the other quantities in (2.26) do not appear at all].

It is worth noting that *apart from the symmetries (2.19), the $t_{A_1\cdots A_n}$ are functionally independent.* This result is essentially a consequence of the independence of the quantities in the basis (2.26), in which $t_A = -D_A{}^0$, $t_{(AB)}$, $\{t_{A_1\cdots A_n}\}_{n=3}^{\infty}$, appear independently. We have to prove in addition that the $t_{[AB]}$ are independent of these and are arbitrary apart from skew-symmetry. This is, indeed, the case since equation (2.20) and $U_A{}^0 = \frac{1}{2}(UU)_{/A} = 0$ imply

$$2t_{[A_1A_2]} = D^\alpha_{A_1}U^\alpha_{A_2} - D^\alpha_{A_2}U^\alpha_{A_1} \equiv K_{A_1A_2} \tag{2.27}$$

The $D_A{}^\alpha$, $U_A{}^\alpha$ appear independently in (2.26) and the K_{AB} are 15 functionally independent quantities with respect to them, as was proved in Section 3.3 of Paper I. (There we wrote $K^{(N)}_{AB}$ instead of K_{AB} here.)

2.4. Definition of the DCMs. The DCMs were defined in the introduction. We now present an equivalent, more constructive, definition. In order to do this we have to know the time derivatives (d/ds) of the quantities (2.26) along the particles' world-lines. This kind of derivative we denote by a dot. We again remind the reader of the fact that all tensor components should be taken with respect to parallel-transported orthogonal tetrads.

Equations (2.3), (2.4), (2.18) imply $\dot{t}_A = 0$. A consequence of this equation and the definition (2.18) is

$$\dot{t}_{A_1\cdots A_n} = 0 \qquad (n = 1, 2, \ldots) \tag{2.28}$$

In particular $\dot{t}_{(AB)} = 0$. By (2.3) $\dot{D}_A{}^i = U_A{}^i$, and by (2.4), (2.7), (2.8), $\dot{U}_A{}^\alpha = -\frac{3}{2}S_{00\alpha\beta}D_A{}^\beta$. Of course, $\dot{S}_{abi_1\cdots i_n} = S_{abi_1\cdots i_n;0}$, but we have to substitute from (A.4) of the Appendix into this equation. So far, the calculation of $\dot{H}_{A_1\cdots A_n}$ is still missing. We first deal with $H_{A_1A_2A_3}$. Its definition by (2.11) and (2.17) and properties of other relevant quantities (in particular, those of $S_{ijklmn\ldots}$) lead to

$$\dot{H}_{A_1A_nA_3} = -6S_{ijkl}U^iU^j_{(A_1}D^k_{A_2}D^l_{A_3)} - \tfrac{3}{2}S_{ijklm}U^iU^jD^k_{A_1}D^l_{A_2}D^m_{A_3}$$

then by (2.17) [making use of $\dot{H}_{A_1\cdots A_n} = (\dot{H}_{A_1A_2A_3})_{/A_4/\cdots/A_n}$]

$$\dot{H}_{A_1\cdots A_n} = -\tfrac{3}{2}S_{abi_1\cdots i_n}U^aU^bD^{i_1}_{A_1}\cdots D^{i_n}_{A_n} + [A(n-2), S(n-1)]_{A_1\cdots A_n}$$

We recall $H_{A_1\cdots A_n} = [A(n-1)]_{A_1\cdots A_n}$. To sum up

$$\dot{s} = 1 \tag{2.29a}$$

$$\dot{D}_A{}^i = U_A{}^i \qquad (U_A{}^0 = 0) \tag{2.29b}$$

$$\dot{U}_A{}^\alpha = -\tfrac{3}{2}S_{00\alpha\beta}D_A{}^\beta \tag{2.29c}$$

$$\dot{t}_{(AB)} = 0 \tag{2.29d}$$

$$\dot{t}_{A_1\cdots A_n} = 0 \qquad (n = 3, 4, \ldots) \tag{2.29e}$$

$$\dot{S}_{abij} = \tfrac{3}{2}S_{abij0} + \tfrac{3}{4}(S_{a0bij} + S_{b0aij}) \tag{2.29f}$$

$$\dot{S}_{abi_1\cdots i_n} = \frac{n(n+1)}{(n-1)(n+2)}S_{abi_1\cdots i_n0} + \frac{n+1}{(n-1)(n+2)} \times (S_{a0bi_1\cdots i_n} + S_{b0ai_1\cdots i_n}) + [Sn]_{abi_1\cdots i_n} \qquad (n = 3, 4, \ldots) \tag{2.29g}$$

$$\dot{H}_{A_1\cdots A_n} = -\tfrac{3}{2}S_{00i_1\cdots i_n}D^{i_1}_{A_1}\cdots D^{i_n}_{A_n} + [A(n-2), S(n-1)]_{A_1\cdots A_n} \qquad (n = 3, 4, \ldots) \tag{2.29h}$$

A function F of a finite number of arguments from (2.26) is a DCM if and only if it satisfies in a certain domain of its arguments the equation

$$\dot{F} \equiv \frac{\partial F}{\partial s} + U_A{}^\alpha\left(\frac{\partial F}{\partial D_A{}^\alpha}\right) - \tfrac{3}{2}S_{00\alpha\beta}D_A{}^\beta\left(\frac{\partial F}{\partial U_A{}^\alpha}\right) + \sum_{k=2}^{\infty}\dot{S}_{abi_1\cdots i_k}\left(\frac{\partial F}{\partial S_{abi_1\cdots i_k}}\right) + \sum_{k=3}^{\infty}\dot{H}_{A_1\cdots A_k}\left(\frac{\partial F}{\partial H_{A_1\cdots A_k}}\right) = 0 \tag{2.30}$$

in which the summations are in fact finite and we have to substitute the relevant expressions from (2.29).

Equation (2.30) is a single equation. But with a function F of a finite number of arguments (as should be), the coefficients in (2.30) contain some quantities of (2.26) that are not among the arguments of F (!). Since these are arbitrary (in their domains), it always follows that equations (2.30) decomposes into a system of equations. In the following we find and characterize all the solutions of these systems.

3. THE DCMs OF EINSTEIN'S THEORY WITH NO RESTRICTIONS

In Section 2.3 we showed that apart from (2.19) the $t_{A_1 \cdots A_n}$ are functionally independent and by (2.28) they are DCMs. In this section we complete the proof of the following assertion.

Apart from the symmetries (2.19), the $t_{A_1 \cdots A_n}$ are functionally independent and they form a basis for the generalized covariant DCMs. Also, they are all covariant in the restricted sense. (The last statement is obvious.)

3.1. DCMs of the First Order, $[F(s, D_A{}^i, U_A{}^\alpha, S_{abi_1i_2}, \ldots, S_{abi_1\cdots i_K})]$. For such functions equation (2.30) reads

$$\frac{\partial F}{\partial s} + U_A{}^\alpha\left(\frac{\partial F}{\partial D_A{}^\alpha}\right) - \tfrac{3}{2}S_{00\alpha\beta}D_A{}^\beta\left(\frac{\partial F}{\partial U_A{}^\alpha}\right) + \sum_{k=2}^{K} S_{abi_1\cdots i_k;0}\left(\frac{\partial F}{\partial S_{abi_1\cdots i_k}}\right) = 0$$

By part (b) of Theorem 2 in the Appendix we may change the $S_{abi_1\cdots i_K;0}$ according to (A.7) without any (other) change of the arguments of F. The above equation is still valid then. Hence $\sigma_{abi_1\cdots i_K;0}(\partial F/\partial S_{abi_1\cdots i_K}) = 0$ for all $\sigma_{abi_1\cdots i_K;0}$ with the symmetries of $S_{abi_1\cdots i_K}$. This means that F has to be a trivial function of the $S_{abi_1\cdots i_K}$. By induction F is a trivial function of all $\{S_{abi_1\cdots i_k}\}_{k=2}^{\infty}$. Equation (2.30) then takes the form

$$\frac{\partial F}{\partial s} + U_A{}^\alpha\left(\frac{\partial F}{\partial D_A{}^\alpha}\right) - \tfrac{3}{2}S_{00\alpha\beta}D_A{}^\beta\left(\frac{\partial F}{\partial U_A{}^\alpha}\right) = 0 \tag{3.1}$$

F is independent of the $S_{00\alpha\beta}$ now, which are arbitrary apart from symmetry in $\alpha\beta$; hence (3.1) is equivalent to

$$\frac{\partial F}{\partial s} + U_A{}^\alpha\left(\frac{\partial F}{\partial D_A{}^\alpha}\right) = 0 \qquad D_A{}^\alpha\left(\frac{\partial F}{\partial U_A{}^\beta}\right) + D_A{}^\beta\left(\frac{\partial F}{\partial U_A{}^\alpha}\right) = 0$$

These equations are equivalent to equations (a) and (b) of Section 3.3 of Paper I, and, as was shown there, they imply that F is independent of s and may depend on the $D_A{}^\alpha$, $U_A{}^\alpha$ only through the $t_{[AB]}$. [Remember (2.27).] Also, its dependence on $D_A{}^0 \equiv -t_A$ is completely arbitrary.

3.2. DCMs of High Orders. Let F be a DCM. Assume that as a function of the basic quantities (2.26) the $H_{A_1\cdots A_n}$ for a certain n ($n \geqslant 3$) occur among its arguments while the $\{H_{A_1\cdots A_k}\}_{k>n}$ do not. By considerations analogous to those made in Section 3.1 [based on (2.29), (2.30), and Theorem 2 of the Appendix], F cannot be a function of the $\{S_{abi_1\cdots i_k}\}_{k\geqslant n}$. Therefore equation (2.30) reads in which we have to substitute $\dot{S}_{abi_1\cdots i_k}$ and $\dot{H}_{A_1\cdots A_k}$ from (2.29).

$$\frac{\partial F}{\partial s} + U_A{}^\alpha\left(\frac{\partial F}{\partial D_A{}^\alpha}\right) - \tfrac{3}{2}S_{00\alpha\beta}D_A{}^\beta\left(\frac{\partial F}{\partial U_A{}^\alpha}\right) + \sum_{k=2}^{n-1} \dot{S}_{abi_1\cdots i_k}\left(\frac{\partial F}{\partial S_{abi_1\cdots i_k}}\right) + \sum_{k=3}^{n} \dot{H}_{A_1\cdots A_k}\left(\frac{\partial F}{\partial H_{A_1\cdots A_k}}\right) = 0 \tag{3.2}$$

In the following we treat linear homogeneous differential equations for the single unknown function F. On these we shall perform operations of crossing and linear combinations. We shall start from equations consisting of part of the terms of (3.2). The coefficients of the derivatives of F in the resulting equations will be polynomials in the quantities (2.26) without the $\{H_{A_1\cdots A_k}\}_{k=n}^{\infty}$, since this is the case in equation (3.2) itself (by induction). Also, in the resulting equations, the coefficients of the derivatives of F with respect to the variables $\{s, D_A{}^\alpha, U_A{}^\alpha, S_{abi_1i_2}, S_{abi_1i_2i_3}, \ldots\}$ will be functions of these quantities themselves only [by induction based on (2.29) and (3.2)]. This fact sometimes enables us to write the resulting equations so that only the terms containing derivatives of F with respect to $\{s, D_A{}^\alpha, U_A{}^\alpha, S_{abi_1i_2}, S_{abi_1i_2i_3}, \ldots\}$ appear explicitly, in the sense that the terms of this type in every new derived equation are determined only by the terms of this type in the preceding (available) equations (induction). We adopt this in the following, and we denote by "$\cdots$" the terms not written explicitly. Also, we adopt the notation of the Appendix, generally. In particular, $\sigma_{abi_1\ldots i_k}$ will denote arbitrary constants with the symmetries (A.8) and

$$\sigma_{abi_1\cdots i_n;0} \equiv \frac{n(n+1)}{(n-1)(n+2)}\,\sigma_{abi_1\cdots i_n0} + \frac{n+1}{(n-1)(n+2)}\,(\sigma_{a0bi_1\cdots i_n} + \sigma_{b0ai_1\cdots i_n}) \tag{3.3}$$

We now return to equation (3.2). By part (b) of Theorem 2 of the Appendix we know that equation (3.2) remains valid if the transformation (A.7) is applied to the $S_{abi_1\cdots i_n}$, $S_{abi_1\cdots i_{n-1};0}$. This implies the equation

$$\sigma_{abi_1\cdots i_{n-1};0}\left(\frac{\partial F}{\partial S_{abi_1\cdots i_{n-1}}}\right) - \tfrac{3}{2}\sigma_{00i_1\cdots i_n}D^{i_1}_{A_1}\cdots D^{i_n}_{A_1}\left(\frac{\partial F}{\partial H_{A_1\cdots A_n}}\right) = 0 \tag{3.4}$$

According to Theorem 2, $\sigma_{00i_1\cdots i_{n-1}0} = [(n-2)/n]\sigma_{00i_1\cdots i_{n-1};0}$, the $\sigma_{abi_1\cdots i_{n-1};0}$ are arbitrary apart from the symmetries of $\sigma_{abi_1\cdots i_{n-1}}$, (A.8), and they are independent of the $\sigma_{00\alpha_1\cdots\alpha_n}$. Therefore, (3.4) is equivalent to the equations

(a$_1$) $$\sigma_{abi_1\cdots i_{n-1}}\left(\frac{\partial F}{\partial S_{abi_1\cdots i_{n-1}}} + \cdots\right) = 0$$

(b) $$\sigma_{00\alpha_1\cdots\alpha_n}D^{\alpha_1}_{A_1}\cdots D^{\alpha_n}_{A_n}\left(\frac{\partial F}{\partial H_{A_1\cdots A_n}}\right) = 0$$

We emphasize that equation (b) is written in explicit form. Applying (a_1) and (A.7) to (3.2) leads to

$$\text{(c)}\quad \frac{\partial F}{\partial s} + U_A{}^\alpha\left(\frac{\partial F}{\partial D_A{}^\alpha}\right) - \tfrac{3}{2}S_{00\alpha\beta}D_A{}^\beta\left(\frac{\partial F}{\partial U_A{}^\alpha}\right) + \sum_{k=2}^{n-2} S_{abi_1\cdots i_k;0}\left(\frac{\partial F}{\partial S_{abi_1\cdots i_k}}\right) + [S(n-1)]_{abi_1\cdots i_{n-1}}\left(\frac{\partial F}{\partial S_{abi_1\cdots i_{n-1}}}\right) + \cdots = 0$$

{In fact, here $[S(n-1)]_{abi_1\cdots i_{n-1}} = 0$; we prefer that form, however, in order that these considerations be applicable in the case of a vacuum.} Now we perform an inductive process which leads to equations (a_2), (a_3), ..., as follows. Each (a_k) has the form

(a_k)

$$\sigma_{abi_1\cdots i_{n-k}}\left\{\frac{\partial F}{\partial S_{abi_1\cdots i_{n-k}}} + \sum_{l=n-k+1}^{n-1} [S(n-1)]^{abi_1\cdots i_{n-k}}_{cdj_1\cdots j_l}\left(\frac{\partial F}{\partial S_{cdj_1\cdots j_l}}\right) + \cdots\right\} = 0$$

Equation (a_{k+1}) is derived by performing [a_k, c]. With the aid of (2.29g) and (3.3) we obtain

$$[a_k, c]\quad \sigma_{abi_1\cdots i_{n-k-1};0}\left(\frac{\partial F}{\partial S_{abi_1\cdots i_{n-k-1}}}\right) + \sigma_{abi_1\cdots i_{n-k}}\sum_{l=n-k}^{n-1} [S(n-1)]^{abi_1\cdots i_{n-k}}_{cdj_1\cdots j_l}\left(\frac{\partial F}{\partial S_{cdj_1\cdots j_l}}\right) + \cdots = 0$$

Since $\sigma_{abi_1\cdots i_{n-k-1};0}$ are arbitrary apart from the symmetries of $\sigma_{abi_1\cdots i_{n-k-1}}$, (Theorem 2), [$a_k$, c] implies an equation of the type (a_{k+1}). Finally for $k = n - 2$ we obtain

$$(a_{n-2})\quad \sigma_{abi_1i_2}\left[\frac{\partial F}{\partial S_{abi_1i_2}} + \sum_{l=3}^{n-1} [S(n-1)]^{abi_1i_2}_{cdj_1\cdots j_l}\frac{\partial F}{\partial S_{cdj_1\cdots j_l}} + \cdots\right] = 0$$

$$[a_{n-2}, c] = (a_{n-1})$$

$$-\tfrac{3}{2}\sigma_{00\alpha\beta}D_A{}^\beta\left(\frac{\partial F}{\partial U_A{}^\alpha}\right) + \sigma_{abi_1i_2}\left\{\sum_{l=2}^{n-1} [S(n-1)]^{abi_1i_2}_{cdj_1\cdots j_l}\left(\frac{\partial F}{\partial S_{cdj_1\cdots j_l}}\right) + \cdots\right\} = 0$$

The $\sigma_{00\alpha\beta}$ are arbitrary apart from symmetry; hence (a_{n-1}) implies

$$\sigma_{00\alpha\beta}\left\{D_A{}^\beta\left(\frac{\partial F}{\partial U_A{}^\alpha}\right) + \sum_{l=2}^{n-1} [S(n-1)]^{\alpha\beta}_{cdj_1\cdots j_l}\left(\frac{\partial F}{\partial S_{cdj_1\cdots j_l}}\right) + \cdots\right\} = 0$$

Therefore

(d)

$$D_A{}^\alpha\left(\frac{\partial F}{\partial U_A{}^\beta}\right) + D_A{}^\beta\left(\frac{\partial F}{\partial U_A{}^\alpha}\right) + \sum_{l=2}^{n-1} [S(n-1)]^{(\alpha\beta)}_{cdj_1\cdots j_l}\left(\frac{\partial F}{\partial S_{cdj_1\cdots j_l}}\right) + \cdots = 0$$

Substitution of (d) in (c) leads to

$$\text{(e)} \quad \frac{\partial F}{\partial s} + U_A{}^{\alpha}\left(\frac{\partial F}{\partial D_A{}^{\alpha}}\right) + \sum_{l=2}^{n-1} [S(n-1)]_{cdj_1\cdots j_l}\left(\frac{\partial F}{\partial S_{cdj_1\cdots j_l}}\right) + \cdots = 0$$

We perform [e, d] = (f), [e, f] = (g),

(g)

$$U_A{}^{\alpha}\left(\frac{\partial F}{\partial D_A{}^{\beta}}\right) + U_A{}^{\beta}\left(\frac{\partial F}{\partial D_A{}^{\alpha}}\right) + \sum_{l=2}^{n-1} [S(n-1)]^{(\alpha\beta)}_{cdj_1\cdots j_l}\left(\frac{\partial F}{\partial S_{cdj_1\cdots j_l}}\right) + \cdots = 0$$

Equation (b) is equivalent to

$$\text{(b*)} \qquad D^{\alpha_1}_{(A_1} \cdots D^{\alpha_n}_{A_n)}\left(\frac{\partial F}{\partial H_{A_1\cdots A_n}}\right) = 0$$

Now we observe that equation (b*) is identical to (c) of Section 3.4 of Paper I, while (g) is, apart from some additional terms, identical with (d) of Section 3.4 of Paper I. Exactly the same process that was carried out there (in Paper I) implies here with the aid of (2.10) that F is independent of the $H_{A_1\cdots A_n}$. By induction F is independent of the $\{H_{A_1\cdots A_k}\}_{k=3}^{\infty}$, and the way to the desired assertion of this section (with the aid of Section 3.1) is open.

4. THE DCMs OF EINSTEIN'S THEORY IN VACUUM

A DCM of Einstein's theory with no restriction is, in particular, a DCM in vacuous space-times. Thus, the set of DCMs in vacuum may be larger than the set of DCMs that hold for all gravitational fields. On the other hand, the vacuum condition, $R_{ij} = 0$, reduces the set of differential quantities, since it introduces, in addition to (2.8), further restrictions on the $S_{abi_1\cdots i_k}$, appearing in the basis (2.26), namely,

$$\eta^{kl} S_{abi_1\cdots i_n kl} + [S(n+1)]_{abi_1\cdots i_n} = 0 \qquad (n = 0, 1, 2, \ldots) \qquad (4.1)$$

(Theorem 3 of the Appendix). In principle these restrictions may make some of the DCMs found in Section 3 trivial. However, what really happens is that *(provided the dimension of space-time is not less than 4) the vacuum condition does not change the set of DCMs at all.*

We outline the proof. Our aim is to show that the $t_{A_1\cdots A_n}$ again form a basis for the DCMs in vacuum. We follow essentially the treatment of Section 3, but from time to time we have to overcome some new difficulties, peculiar to this situation. Usually this means that we have to use Theorem 4 in the Appendix rather than Theorem 2, and, also, the constants $\sigma_{abi_1\cdots i_k}$ introduced in Section 3.2 have to satisfy (A.19) in addition to (A.8).

4.1. DCMs of the First Order, $[F(s, D_A^i, U_A^\alpha, S_{abi_1i_2}, \ldots, S_{abi_1\cdots i_K})]$. We follow the argumentation of Section 3.1 and we find that F has to satisfy $\sigma_{abi_1\cdots i_K;0}(\partial F/\partial S_{abi_1\cdots i_K}) = 0$, for all $\sigma_{abi_1\cdots i_K;0}$ obtained by (3.3). Now, by Theorem 4 of the Appendix, given $\{s, D_A^i, U_A^\alpha, S_{abi_1i_2}, \ldots, S_{abi_1\cdots i_{K-1}}\}$, the freedom in $S_{abi_1\cdots i_K}$ is of the type $S_{abi_1\cdots i_K} \to S_{abi_1\cdots i_K} + \sigma_{abi_1\cdots i_K}$. It follows that the above equation implies that F is a trivial function of $S_{abi_1\cdots i_K}$ if the $\sigma_{abi_1\cdots i_K;0}$ that satisfy the symmetries of $\sigma_{abi_1\cdots i_K}$ are otherwise arbitrary. This is assured by Theorem 5 provided the dimension of space-time is 4 at least. In this case we obtain by induction that F cannot be a nontrivial function of the $\{S_{abi_1\cdots i_k}\}_{k=2}^\infty$. Then F satisfies (3.1), with $S_{00\alpha\beta}$ arbitrary apart from symmetry and $S_{00\alpha\alpha} = 0$. This equation is identical to equations (a), (b*) of Section 4.1 of Paper I, which imply that F is independent of s and it may depend on the D_A^α, U_A^α only through the $t_{[AB]}$ as expected. [Remember (2.27).]

4.2. DCMs of High Orders. We follow the argumentation of Section 3.2 and adopt the conventions introduced there. Assume that F does depend on the $H_{A_1\cdots A_n}$ for a certain n ($n \geqslant 3$) and does not depend on the $\{H_{A_1\cdots A_k}\}_{k>n}$. By considerations analogous to those done in Section 4.1 F cannot be a function of the $\{S_{abi_1\cdots i_k}\}_{k\geqslant n}$, provided the dimension of space-time is at least 4. Then F satisfies equations (3.2) and (3.4). By a process similar to that performed in Section 3.2 (but using Theorem 4 rather than Theorem 2) it follows, in space-time of dimension 4 at least, that F satisfies

$$\text{(a)} \quad \frac{\partial F}{\partial s} + U_A^\alpha\left(\frac{\partial F}{\partial D_A^\alpha}\right) + \sum_{l=2}^{n-1} [S(n-1)]_{abi_1\cdots i_l}\left(\frac{\partial F}{\partial S_{abi_1\cdots i_l}}\right) + \cdots = 0$$

$$\text{(b)} \quad \sigma_{00\alpha\beta}\left\{D_A^\beta\left(\frac{\partial F}{\partial U_A^\alpha}\right) + \sum_{l=2}^{n-1} [S(n-1)]^{\alpha\beta}_{abi_1\cdots i_l}\left(\frac{\partial F}{\partial S_{abi_1\cdots i_l}}\right) + \cdots\right\} = 0$$

where $\sigma_{00\alpha\beta}$ is arbitrary apart from symmetry and $\sigma_{00\alpha\alpha} = 0$. By the remark following Theorem 4, equation (3.4) implies

$$\text{(c)} \qquad \sigma_{00\alpha_1\cdots\alpha_n} D_{A_1}^{\alpha_1}\cdots D_{A_n}^{\alpha_n}\left(\frac{\partial F}{\partial H_{A_1\cdots A_n}}\right) = 0$$

for all symmetric $\sigma_{00\alpha_1\cdots\alpha_n}$ that satisfy $\sigma_{00\alpha_1\cdots\alpha_{n-2}\mu\mu} = 0$.

Equations (a), (b), and (c) are similar to the equations obtained in Section 4.2 of Paper I [(c) is even identical to (c) there]. The same process carried out in Section 4.2 of Paper I implies here, with the aid of (2.10), that F cannot be a nontrivial function of the $H_{A_1\cdots A_n}$. By induction F is independent of the $\{H_{A_1\cdots A_k}\}_{k=3}^\infty$ and the way to the desired assertion of this section is open.

5. SOME CONCLUDING REMARKS

By means of slight modifications of the foregoing proofs we can generalize the results obtained for Einstein's theory with no restrictions to the classes of

Riemann's spaces (signature arbitrary) of dimension n, $n \geqslant 2$, and the results obtained for Einstein's theory in vacuum to the classes of vacuous Riemann's spaces ($R_{ij} = 0$) (signature arbitrary) of dimension n, $n \geqslant 4$. The DCMs for these classes are all covariant, and the $\{t_{A_1 \cdots A_k}\}_{k=1}^{\infty}$, defined by (2.18), form a basis for them. The existence of these DCMs is implied by the fact that the t_A are DCMs (Section 2.4). Since the last fact in Einstein's theory is simply the conservation of simultaneity of close clocks (Enosh and Kovetz, 1972), we may say that the existence of every DCM in general relativity is implied by this property. The fact that all the DCMs are covariant in the narrow sense means, in particular, that it is not possible by (local) measurements of differential quantities to determine the orientation of a laboratory relative to Fermi-transported axes (a physical transport law), that is, we cannot endow this differential law with any global meaning. This property is common to Einstein's and Newton's theories of gravitation.

In order to enlarge the analogy between Einstein's and Newton's theories of gravitation we wish to find any (formal, at least) correspondence between the DCMs in both the theories. We notice that in Einstein's theory the DCMs usually depend on the zero adjustment of the time along the clocks; that is, the DCMs usually depend on transformations of parameters of the type

$$s \to s' = s + f(d), \qquad d \to d' = d$$

where f is an arbitrary function of d. Such quantities are artificial in the framework of Newton's theory. We shall look for those DCMs in Einstein's theory that do *not* depend on these transformations. Such a transformation implies

$$U^i \to U'^i = U^i, \qquad D_A{}^i \to D'^i_A = D_A{}^i - U^i f_{/A}, \qquad \frac{\partial}{\partial d^A} = \frac{\partial}{\partial d^A} - f_{/A}\frac{\partial}{\partial s}$$

Hence, by (2.18)

$$t_{A_1 \cdots A_n} \to t'_{A_1 \cdots A_n} = t_{A_1 \cdots A_n} + f_{A_1 \cdots A_n}$$

where $f_{A_1 \cdots A_n} = f_{(A_1 \cdots A_n)} = f_{/A_1/ \cdots /A_n}$ are arbitrary apart from symmetry. Since $t_{A_1 \cdots A_n} = t_{A_1(A_2 \cdots A_n)}$, we may replace, according to Lemma 1 in the Appendix of Paper I, every $t_{A_1 \cdots A_n}$ by the pair

$$\langle t_{(A_1 \cdots A_n)}, 2t_{[A_1 A_2]A_3 \cdots A_n} \equiv K_{A_1 \cdots A_n} \rangle$$

The transformation above implies

$$t_{(A_1 \cdots A_n)} \to t'_{(A_1 \cdots A_n)} = t_{(A_1 \cdots A_n)} + f_{A_1 \cdots A_n}, \qquad K_{A_1 \cdots A_n} \to K'_{A_1 \cdots A_n} = K_{A_1 \cdots A_n}$$

Hence the $K_{A_1 \cdots A_n}$ form a basis for the desired DCMs. We recognize the analogy with the $K^{(N)}_{A_1 \cdots A_n}$ of Newton's theory (Paper I). In particular the

$K_{A_1\cdots A_n}$ and the $K^{(N)}_{A_1\cdots A_n}$ have the same symmetry properties; in a local inertial rest frame $K^{(N)}_{AB} = K_{AB}$ (this is incorrect for higher orders of K's); and

$$K_{A_1\cdots A_n} = K_{A_1A_2/A_3/\cdots/A_n}, \qquad K^{(N)}_{A_1\cdots A_n} = K^{(N)}_{A_1A_2/A_3/\cdots/A_n}$$

The classes of vacuous Riemannian spaces (with any signature) of dimension 2 or 3 do not constitute a real problem, since these spaces are *flat* (!). Finding the DCMs is now trivial, since all equations of evolution for the differential quantities are explicitly solvable [the basis (2.9) is preferable]. We shall not do this here. We note, however, that the set of DCMs is much larger, and it includes generalized covariant quantities that are not covariant. [In special relativity it is possible to fix the orientation of a nonrotating laboratory by means of local measurements only, since by (2.29c), for example, the three-vectors $U_A{}^\alpha$ are constant with respect to parallel-transported axes. This fact is to be expected, since free particles move along straight lines in Minkowski coordinates of a flat space; hence, four free particles that are moving in parallel—common four-velocity—and are, respectively, located at the origin and at three points on the spatial axes of their common rest frame, remain in their relative positions and fix parallel-transported axes with time. Our discussion demonstrates that such constructions are impossible in more general situations of general relativity.]

APPENDIX: PROPERTIES OF THE $S_{abi_1\cdots i_k}$

It is well known that the covariant derivatives of Riemann's tensor form a complete set for the differential concomitants of the Riemannian metric. In other words, the components of the covariant derivatives of Riemann's tensor with respect to a given orthonormal tetrad at a given event determine and are determined by the derivatives of the metric components at this event in the normal coordinates (Schouten, 1954, p. 155) with origin at this event and axes coinciding with the given tetrad. Moreover, the $\{g_{ab/i_1/\cdots/i_k}\}_{k\leqslant K}$ determine and are determined by the $\{R_{abi_1i_2;i_3;\ldots;i_k}\}_{k\leqslant K}$. The trouble is that the $\{R_{abi_1i_2;i_3;\ldots;i_k}\}$ are not independent of each other. We adopt a proposition by Penrose for a complete and functionally independent set for these quantities. We define, after Synge (Synge, 1960, p. 54), the symmetric curvature tensor

$$S_{ijkl} \equiv \tfrac{1}{3}(R_{ikjl} + R_{jkil}) \Leftrightarrow R_{ijkl} = S_{ikjl} - S_{jkil} \tag{A.1}$$

and after Penrose (1960),

$$S_{abi_1\cdots i_n} \equiv S_{ab(i_1i_2;i_3;\ldots;i_n)} \qquad (n = 2, 3, 4, \ldots) \tag{A.2}$$

According to Penrose, *apart from the symmetries*

$$S_{abi_1\cdots i_n} = S_{(ab)(i_1\cdots i_n)}, \qquad S_{a(bi_1\cdots i_n)} = 0 \qquad (n = 2, 3, \ldots) \tag{A.3}$$

the $\{S_{abi_1\cdots i_k}\}_{k=2}^{\infty}$ *are functionally independent and they form a complete set for the covariant derivatives of* R_{ijkl}.

The symmetries (A.3) can be proven by induction as consequences of (A.1), (A.2), the symmetries of R_{ijkl} and the relation $S_{abi_1\cdots i_{n+1}} = S_{ab(i_1\cdots i_n;i_{n+1})}$ implied by (A.2). In order to prove that the $\{S_{abi_1\cdots i_k}\}_{k=2}^{\infty}$ form a complete set and, in particular, for other applications, too, we need the following lemmas.

Lemma 1.

$$S_{abi_1\cdots i_{n-1}[i_n;i_{n+1}]} = -\frac{n+1}{(n-1)(n+2)} \times (S_{a[i_n i_{n+1}]bi_1\cdots i_{n-1}} + S_{b[i_n i_{n+1}]ai_1\cdots i_{n-1}}) + \{n\}_{abi_1\cdots i_{n+1}} \qquad (n = 2, 3, \ldots)$$

where $\{n\}_{abi_1\cdots i_{n+1}}$ denote certain polynomials in the $\{R_{abi_1i_2;i_3;\ldots;i_k}\}_{k\leqslant n}$.

We leave the proof to the reader; nevertheless we offer two methods. One is by expressing both sides of the desired equation by means of the covariant derivatives of R_{ijkl} and applying the symmetries of R_{ijkl} and the Bianchi identity to their "leading terms." The second is by expressing both sides of the desired equation by means of partial derivatives of g_{ij} and, again, treating the "leading terms" only.

Once we have Lemma 1 (here), by equations (A.2) and (A.3) and by Lemma 1 of the Appendix of Paper I we easily obtain the following.

Lemma 2.

$$S_{abi_1\cdots i_n;i_{n+1}} = \frac{n(n+1)}{(n-1)(n+2)} S_{abi_1\cdots i_{n+1}} + \frac{2(n+1)}{(n-1)(n+2)} S_{i_{n+1}(ab)i_1\cdots i_n} + \{n\}_{abi_1\cdots i_{n+1}} \qquad (n = 2, 3, \ldots)$$

Lemma 2 and (A.1) obviously imply by induction that the $\{S_{abi_1\cdots i_k}\}_{k=2}^{\infty}$ form a complete set for the covariant derivatives of R_{ijkl}; moreover, *the* $\{S_{abi_1\cdots i_k}\}_{k\leqslant K}$ *determine and are determined by the* $\{R_{abi_1i_2;i_3;\ldots;i_k}\}_{k\leqslant K}$.

In order to prove that there are no restrictions on the $\{S_{abi_1\cdots i_k}\}_{k=2}^{\infty}$, except for those of (A.3), we notice that it is not difficult to show that the number of independent quantities among the $\{g_{ab/i_1/\cdots/i_k}\}_{k\leqslant K}$ in normal coordinates equals the number of independent quantities among the $\{S_{abi_1\cdots i_k}\}_{k\leqslant K}$ restricted by (A.3). Since the latter quantities determine the previous ones it is not possible that the $\{S_{abi_1\cdots i_k}\}_{k\leqslant K}$ are restricted any further.

Now we may introduce classifications of functions of the $\{S_{abi_1\cdots i_k}\}_{k=2}^{\infty}$ as follows: *Such a function is of S order n if it is a nontrivial function of the*

$\{S_{abi_1\cdots i_n}\}$ *and does not depend on the* $\{S_{abi_1\cdots i_k}\}_{k>n}$. *Also, we denote by* $[Sn]$, *usually accompanied by indices, any polynomial in the* $\{S_{abi_1\cdots i_k}\}_{k=2}^{n}$. (Its S order is n at most.) Lemma 2 now takes the form of the following theorem:

Theorem 1.

$$S_{abi_1\cdots i_n;i_{n+1}} = \frac{n(n+1)}{(n-1)(n+2)} S_{abi_1\cdots i_{n+1}} + \frac{2(n+1)}{(n-1)(n+2)} S_{i_{n+1}(ab)i_1\cdots i_n} + [Sn]_{abi_1\cdots i_{n+1}} \qquad (n = 2, 3, \ldots) \tag{A.4}$$

For $n = 2$, $[S2]_{abi_1i_2i_3} = 0$ in (A.4).

Checking carefully (A.4) for $i_{n+1} = 0$, and making use of (A.3), immediately lead to the following.

Lemma 3.

$$S_{00i_1\cdots i_n0} = \frac{n-1}{n+1} S_{00i_1\cdots i_n;0} + [Sn]_{i_1\cdots i_n} \tag{A.5a}$$

$$S_{0\alpha i_1\cdots i_{n-1}00} = \frac{(n+2)(n-1)}{(n+1)^2} S_{0\alpha i_1\cdots i_{n-1}0;0} - \frac{n-1}{(n+1)^2} S_{00\alpha i_1\cdots i_{n-1};0} + [Sn]_{\alpha i_1\cdots i_{n-1}} \tag{A.5b}$$

$$S_{0\alpha\gamma_1\cdots\gamma_n0} = \frac{(n+2)(n-1)}{(n+1)^2} S_{0\alpha\gamma_1\cdots\gamma_n;0} - \frac{1}{n+1} S_{00\alpha\gamma_1\cdots\gamma_n} + [Sn]_{\alpha\gamma_1\cdots\gamma_n} \tag{A.5c}$$

$$S_{\alpha\beta i_1\cdots i_{n-2}000} = \frac{(n+2)(n-1)}{(n+1)n} S_{\alpha\beta i_1\cdots i_{n-2}00;0} - \frac{2(n+2)(n-1)}{(n+1)^2 n} S_{0(\alpha\beta)i_1\cdots i_{n-2}0;0} + \frac{2(n-1)}{(n+1)^2 n} S_{00\alpha\beta i_1\cdots i_{n-2};0} + [Sn]_{\alpha\beta i_1\cdots i_{n-2}} \tag{A.5d}$$

$$S_{\alpha\beta\gamma_1\cdots\gamma_{n-1}00} = \frac{(n+2)(n-1)}{(n+1)n} S_{\alpha\beta\gamma_1\cdots\gamma_{n-1}0;0} - \frac{2(n+2)(n-1)}{(n+1)^2 n} S_{0(\alpha\beta)\gamma_1\cdots\gamma_{n-1};0} + \frac{2}{(n+1)n} S_{00\alpha\beta\gamma_1\cdots\gamma_{n-1}} + [Sn]_{\alpha\beta\gamma_1\cdots\gamma_{n-1}} \tag{A.5e}$$

$$S_{\alpha\beta\gamma_1\cdots\gamma_n 0} = \frac{(n+2)(n-1)}{(n+1)n} S_{\alpha\beta\gamma_1\cdots\gamma_n;0} - \frac{2}{n} S_{0(\alpha\beta)\gamma_1\cdots\gamma_n} + [Sn]_{\alpha\beta\gamma_1\cdots\gamma_n} \tag{A.5f}$$

We arrive now at the following important theorem.

Theorem 2. Assume that at a given event an orthonormal tetrad is given and we express all the following tensors by means of their components with respect to this tetrad. Assume, further, that the $\{S_{abi_1\cdots i_k}\}_{k\leqslant n}$ are fixed (given). Then we have the following.

(a) Equation (A.4) for $i_{n+1} = 0$ determines a one-to-one linear transformation of the quantities $\{S_{abi_1\cdots i_{n+1}}\}$ which satisfy (A.3) *onto* the 4-tuples $\{\langle S_{\alpha\beta\gamma_1\cdots\gamma_{n+1}}, S_{0\beta\gamma_1\cdots\gamma_{n+1}}, S_{00\gamma_1\cdots\gamma_{n+1}}, S_{abi_1\cdots i_n;0}\rangle\}$ of quantities which satisfy

$$S_{\alpha\beta\gamma_1\cdots\gamma_{n+1}} = S_{(\alpha\beta)(\gamma_1\cdots\gamma_{n+1})}, \qquad S_{\alpha(\beta\gamma_1\cdots\gamma_{n+1})} = 0,$$
$$S_{0\beta\gamma_1\cdots\gamma_{n+1}} = S_{0\beta(\gamma_1\cdots\gamma_{n+1})}, \qquad S_{0(\beta\gamma_1\cdots\gamma_{n+1})} = 0,$$
$$S_{00\gamma_1\cdots\gamma_{n+1}} = S_{00(\gamma_1\cdots\gamma_{n+1})}, \qquad S_{abi_1\cdots i_n;0} = S_{(ab)(i_1\cdots i_n);0},$$
$$S_{a(bi_1\cdots i_n);0} = 0 \tag{A.6}$$

This transformation reduces to the identity for the $\{S_{\alpha\beta\gamma_1\cdots\gamma_{n+1}}, S_{0\beta\gamma_1\cdots\gamma_{n+1}}, S_{00\gamma_1\cdots\gamma_{n+1}}\}$. [In particular, apart from the symmetries $S_{abi_1\cdots i_n;0} = S_{(ab)(i_1\cdots i_n);0}$; $S_{a(bi_1\cdots i_n);0} = 0$; the $\{S_{abi_1\cdots i_n;0}\}$, by themselves, are arbitrary.] The inverse transformation is given partially by (A.5) (Lemma 3) and is completed by the symmetries (A.3) of $S_{abi_1\cdots i_{n+1}}$.

(b) The freedom in the available $\{\langle S_{abi_1\cdots i_{n+1}}, S_{abi_1\cdots i_n;0}\rangle\}$ is of the type

$$S_{abi_1\cdots i_{n+1}} \to S_{abi_1\cdots i_{n+1}} + \sigma_{abi_1\cdots i_{n+1}},$$
$$S_{abi_1\cdots i_n;0} \to S_{abi_1\cdots i_n;0} + \sigma_{abi_1\cdots i_n;0} \tag{A.7}$$

where

$$\sigma_{abi_1\cdots i_{n+1}} = \sigma_{(ab)(i_1\cdots i_{n+1})}, \qquad \sigma_{a(bi_1\cdots i_{n+1})} = 0 \tag{A.8}$$

$$\sigma_{abi_1\cdots i_n;0} = \frac{n(n+1)}{(n-1)(n+2)}\sigma_{abi_1\cdots i_n 0} + \frac{2(n+1)}{(n-1)(n+2)}\sigma_{0(ab)i_1\cdots i_n} \tag{A.9}$$

Also, equation (A.9) determines a one-to-one linear homogeneous transformation of the quantities $\{\sigma_{abi_1\cdots i_{n+1}}\}$ which satisfy (A.8) *onto* the 4-tuples $\{\langle\sigma_{\alpha\beta\gamma_1\cdots\gamma_{n+1}}, \sigma_{0\beta\gamma_1\cdots\gamma_{n+1}}, \sigma_{00\gamma_1\cdots\gamma_{n+1}}, \sigma_{abi_1\cdots i_n;0}\rangle\}$ of quantities that satisfy

$$\begin{aligned}
\sigma_{\alpha\beta\gamma_1\cdots\gamma_{n+1}} &= \sigma_{(\alpha\beta)(\gamma_1\cdots\gamma_{n+1})}, & \sigma_{\alpha(\beta\gamma_1\cdots\gamma_{n+1})} &= 0,\\
\sigma_{0\beta\gamma_1\cdots\gamma_{n+1}} &= \sigma_{0\beta(\gamma_1\cdots\gamma_{n+1})}, & \sigma_{0(\beta\gamma_1\cdots\gamma_{n+1})} &= 0,\\
\sigma_{00\gamma_1\cdots\gamma_{n+1}} &= \sigma_{00(\gamma_1\cdots\gamma_{n+1})}, & \sigma_{abi_1\cdots i_n;0} &= \sigma_{(ab)(i_1\cdots i_n);0},\\
& & \sigma_{a(bi_1\cdots i_n);0} &= 0
\end{aligned} \tag{A.10}$$

This transformation reduced to the identity for the $\{\sigma_{\alpha\beta\gamma_1\cdots\gamma_{n+1}}, \sigma_{0\beta\gamma_1\cdots\gamma_{n+1}}, \sigma_{00\gamma_1\cdots\gamma_{n+1}}\}$. (In particular, apart from the symmetries $\sigma_{abi_1\cdots i_n;0} = \sigma_{(ab)(i_1\cdots i_n);0}$, $\sigma_{a(bi_1\cdots i_n);0} = 0$; the $\{\sigma_{abi_1\cdots i_n;0}\}$, by themselves, are arbitrary.) The inverse transformation is determined by the symmetries (A.8) and by the equations

$$\sigma_{00i_1\cdots i_n 0} = \frac{n-1}{n+1}\sigma_{00i_1\cdots i_n;0} \tag{A.11a}$$

$$\sigma_{0\alpha i_1\cdots i_{n-1}00} = \frac{(n+2)(n-1)}{(n+1)^2}\sigma_{0\alpha i_1\cdots i_{n-1}0;0} - \frac{n-1}{(n+1)^2}\sigma_{00\alpha i_1\cdots i_{n-1};0} \tag{A.11b}$$

$$\sigma_{0\alpha\gamma_1\cdots\gamma_n 0} = \frac{(n+2)(n-1)}{(n+1)^2}\sigma_{0\alpha\gamma_1\cdots\gamma_n;0} - \frac{1}{n+1}\sigma_{00\alpha\gamma_1\cdots\gamma_n} \tag{A.11c}$$

$$\begin{aligned}
\sigma_{\alpha\beta i_1\cdots i_{n-2}000} = {} & \frac{(n+2)(n-1)}{(n+1)n}\sigma_{\alpha\beta i_1\cdots i_{n-2}00;0}\\
& - \frac{2(n+2)(n-1)}{(n+1)^2 n}\sigma_{0(\alpha\beta)i_1\cdots i_{n-2}0;0}\\
& + \frac{2(n-1)}{(n+1)^2 n}\sigma_{00\alpha\beta i_1\cdots i_{n-2};0}
\end{aligned} \tag{A.11d}$$

$$\begin{aligned}
\sigma_{\alpha\beta\gamma_1\cdots\gamma_{n-1}00} = {} & \frac{(n+2)(n-1)}{(n+1)n}\sigma_{\alpha\beta\gamma_1\cdots\gamma_{n-1}0;0}\\
& - \frac{2(n+2)(n-1)}{(n+1)^2 n}\sigma_{0(\alpha\beta)\gamma_1\cdots\gamma_{n-1};0}\\
& + \frac{2}{n(n+1)}\sigma_{00\alpha\beta\gamma_1\cdots\gamma_{n-1}}
\end{aligned} \tag{A.11e}$$

$$\sigma_{\alpha\beta\gamma_1\cdots\gamma_n 0} = \frac{(n+2)(n-1)}{(n+1)n}\sigma_{\alpha\beta\gamma_1\cdots\gamma_n;0} - \frac{2}{n}\sigma_{0(\alpha\beta)\gamma_1\cdots\gamma_n} \tag{A.11f}$$

Proof. The transformation determined by (A.4) for $i_{n+1} = 0$ is one-to-one since, according to Lemma 3 it has an inverse transformation determined by (A.5). Starting with (A.5), this inverse transformation can be extended linearly to be defined over the whole space of the 4-tuples $\langle S_{\alpha\beta\gamma_1\cdots\gamma_{n+1}}, S_{0\beta\gamma_1\cdots\gamma_{n+1}}, S_{00\gamma_1\cdots\gamma_{n+1}}, S_{abi_1\cdots i_n;0}\rangle$ with the symmetries (A.6). It is easy to see that this extension is uniquely determined by (A.5) and that it constitutes a one-to-one transformation. Also, it maps its (extended) domain of definition into the space of the $S_{abi_1\cdots i_{n+1}}$ with the symmetries (A.3). The last step is a little bit tedious and it constitutes the main part of the proof; but it follows from a straightforward calculation exhausting the examination of all the cases of (A.3). Moreover, this transformation is *onto* the space of the $S_{abi_1\cdots i_{n+1}}$ satisfying (A.3), since it extends an inverse transformation of a transformation defined over the whole of this space. This situation obviously implies that the transformation determined by (A.4) is *onto* the above-mentioned space of the 4-tuples. This accomplishes the proof of part (a) of Theorem 2. Part (b) is a direct consequence of part (a).

In the remainder of the Appendix we shall characterize the additional restrictions over the $\{S_{abi_1\cdots i_k}\}_{k=2}^{\infty}$, implied by the further claim that space-time is vacuous; that is,

$$R_{ij} = 0 \tag{A.12}$$

The restrictions on the covariant derivatives of Riemann's tensor at one event due to the vacuum condition are given by $R_{ab;i_1;\ldots;i_n} = 0$ $(n = 0, 1, 2, \ldots)$. *From now on all the tensors components are understood to be taken along a given orthonormal tetrad.* Since $\frac{2}{3}R_{ab} = \eta^{mn}S_{abmn}$, we get the following.

Lemma 4. The extra restrictions on the $\{S_{abi_1\cdots i_k}\}_{k=2}^{\infty}$ at a point due to the vacuum condition are

$$\eta^{mn}S_{abmn;i_1;\ldots;i_k} = 0 \qquad (k = 0, 1, 2, \ldots) \tag{A.13}$$

We would like to express the vacuum restrictions by means of the $\{S_{abi_1\cdots i_k}\}_{k=2}^{\infty}$ only. For our applications the following theorem is sufficient, however.

Theorem 3. The vacuum condition, (A.13), is equivalent to a certain infinite system of algebraic equations of the form

$$\eta^{mn}S_{abi_1\cdots i_k mn} + [S(k+1)]_{abi_1\cdots i_k} = 0 \qquad (k = 0, 1, 2, \ldots) \tag{A.14}$$

(These equations are not linear, but are quasi-linear.)

Proof. Since $\eta^{mn}S_{abmn;i_1;\ldots;i_k} = S_{ab}{}^{t}{}_{t;i_1;\ldots;i_k}$ it is easy to show by induction that (A.13) is equivalent to

$$\eta^{mn}S_{abmn;(i_1;\ldots;i_k)} = 0 \qquad (k = 0, 1, 2, \ldots) \tag{A.15}$$

Next, Theorem 1 implies, by induction,

$$\eta^{mn}S_{abmn;i_1} = \eta^{mn}(\tfrac{3}{2}S_{abi_1mn} + \tfrac{3}{2}S_{i_1(ab)mn}) \tag{A.16a}$$

$$\begin{aligned}\eta^{mn}S_{abmn;(i_1;\dots;i_k)} = \eta^{mn}\Bigg[&\frac{3(k+1)}{k+3}S_{abi_1\cdots i_kmn} \\ &+ \frac{(k-1)(k+1)(k+6)}{8(k+3)}S_{(i_1\cdots i_k)abmn} \\ &+ \frac{3k(k+1)}{2(k+3)}(S_{a(i_1\cdots i_k)bmn} + S_{b(i_1\cdots i_k)amn})\Bigg] \\ &+ [S(k+1)]_{abi_1\cdots i_k} \qquad (k = 2, 3, \dots)\end{aligned} \tag{A.16b}$$

Now, (A.15) for $k = 0$ is exactly the same as (A.14) for $k = 0$. Further, (A.15) for $k = 1$, $\eta^{mn}S_{abmn;i_1} = 0$, implies $\eta^{mn}(S_{abmn;i_1} - S_{ai_1mn;b}) = 0$. This, with the aid of (A.16a), implies $\eta^{mn}S_{a[bi_1]mn} = 0$. Therefore, $\eta^{mn}S_{abi_1mn}$ is symmetric in all the free indices. Now, again $\eta^{mn}S_{abmn;i_1} = 0$, with the aid of (A.16a), implies $\eta^{mn}S_{abi_1mn} = 0$, which is exactly (A.14) for $k = 1$. Conversely, the last equation implies, with the aid of (A.16a), $\eta^{mn}S_{abmn;i_1} = 0$. We continue: (A.15) for another k ($k = 2, 3, \dots$), is equivalent to the vanishing of the right-hand side of (A.16b). We regard this equation as a system of linear nonhomogeneous equations for the unknowns $\{\eta^{mn}S_{abi_1\cdots i_kmn}\}$. The number of the equations of this system is equal to the number of the unknowns, since both the $\{\eta^{mn}S_{abi_1\cdots i_kmn}\}$ and the $\{[S(k+1)]_{abi_1\cdots i_k}\}$ there are symmetric in the $\{a, b\}$ and in the $\{i_1, \dots, i_k\}$. We shall prove that this system is regular. This would mean that it is solvable and therefore is equivalent to a system of solutions $\eta^{mn}S_{abi_1\cdots i_kmn} = [S(k+1)]_{abi_1\cdots i_k}$, which is indeed equivalent to (A.14). In order to prove regularity of the system (A.16b) it is sufficient to show that its associated homogeneous system {in which $[S(k+1)]_{abi_1\cdots i_k} = 0$}, possesses only the trivial solution. Indeed, setting $[S(k+1)]_{abi_1\cdots i_k} = 0$ in (A.16b) and subtracting the equation obtained by interchanging the indices b and i_1 from it results in

$$\eta^{mn}\Bigg[(k+6)S_{a[bi_1]i_2\cdots i_kmn} + (11k-6) \times \left(S_{a[bi_1]i_2\cdots i_kmn} + \sum_{r=2}^{k} S_{i_r[bi_1]i_2\cdots\hat{i}_r\cdots i_kamn}\right)\Bigg] = 0$$

Symmetrization of the indices $\{a, i_2, \dots, i_k\}$ in this equation leads to

$$(11k^2 - 5k + 6)\eta^{mn}\left(S_{a[bi_1]i_2\cdots i_kmn} + \sum_{r=2}^{k} S_{i_r[bi_1]i_2\cdots\hat{i}_r\cdots i_kamn}\right) = 0$$

Since $11k^2 - 5k + 6 > 0$ we derive by substitution of the last equation in the preceding one $\eta^{mn}S_{a[bi_1]i_2\cdots i_kmn} = 0$. Hence $\eta^{mn}S_{abi_1\cdots i_kmn}$ is symmetric in all its (free) indices. Applying this again to the homogeneous system immediately implies $\eta^{mn}S_{abi_1\cdots i_kmn} = 0$. This completes the proof of Theorem 3.

An obvious consequence of (A.14) and the symmetries (A.3) is the following.

Lemma 5. The vacuum condition implies equations of the form

$$\eta^{mn} S_{amni_1\cdots i_k} + [Sk]_{ai_1\cdots i_k} = 0 \qquad (k = 1, 2, \ldots) \quad \text{(A.17a)}$$

$$\eta^{mn} S_{mni_1\cdots i_k} + [S(k-1)]_{i_1\cdots i_k} = 0 \qquad (k = 2, 3, \ldots) \quad \text{(A.17b)}$$

From Theorem 2 we know that it is possible to express every $S_{abi_1\cdots i_{n+1}}$ by means of $S_{\alpha\beta\gamma_1\cdots\gamma_{n+1}}$, $S_{0\beta\gamma_1\cdots\gamma_{n+1}}$, $S_{00\gamma_1\cdots\gamma_{n+1}}$, $S_{abi_1\cdots i_n;0}$ and quantities of lower S order; we even know the form of these expressions [given mainly by (A.5)]. Applying this carefully to Theorem 3 implies the following.

Lemma 6. An equivalent form to the vacuum condition, (A.14), is the following system of equations.

$$\eta^{mn} S_{abi_1\cdots i_{k-2}mn;0} + [Sk]_{abi_1\cdots i_{k-2}} = 0 \qquad \text{(A.18a)}$$

$$\frac{k-1}{k+1} S_{00\gamma_1\cdots\gamma_{k-1}0;0} - S_{00\gamma_1\cdots\gamma_{k-1}\mu\mu} + [Sk]_{\gamma_1\cdots\gamma_{k-1}} = 0 \quad \text{(A.18b)}$$

$$\frac{(k+2)(k-1)}{(k+1)^2} S_{0\beta\gamma_1\cdots\gamma_{k-1}0;0} - \frac{k-1}{(k+1)^2} S_{00\beta\gamma_1\cdots\gamma_{k-1};0} - S_{0\beta\gamma_1\cdots\gamma_{k-1}\mu\mu} + [Sk]_{\beta\gamma_1\cdots\gamma_{k-1}} = 0 \quad \text{(A.18c)}$$

$$\frac{(k-1)(k+2)}{(k+1)k} S_{\alpha\beta\gamma_1\cdots\gamma_{k-1}0;0} - \frac{2(k-1)(k+2)}{k(k+1)^2} S_{0(\alpha\beta)\gamma_1\cdots\gamma_{k-1};0} + \frac{2}{k(k+1)} S_{00\alpha\beta\gamma_1\cdots\gamma_{k-1}} - S_{\alpha\beta\gamma_1\cdots\gamma_{k-1}\mu\mu} + [Sk]_{\alpha\beta\gamma_1\cdots\gamma_{k-1}} = 0 \quad \text{(A.18d)}$$

The following important theorem is analogous to Theorem 2 in the case of vacuum.

Theorem 4. The assumptions made in Theorem 2 and the further assumptions that space-time is vacuous ($R_{ij} = 0$) lead to the following.

(a) Equation (A.4) for $i_{n+1} = 0$ determines a one-to-one linear transformation of the quantities $\{S_{abi_1\cdots i_{n+1}}\}$ which satisfy (A.3) and (A.14) *onto* the 4-tuples $\{\langle S_{\alpha\beta\gamma_1\cdots\gamma_{n+1}}, S_{0\beta\gamma_1\cdots\gamma_{n+1}}, S_{00\gamma_1\cdots\gamma_{n+1}}, S_{abi_1\cdots i_n;0}\rangle\}$ of quantities that satisfy (A.6) and (A.18). The inverse transformation is determined by (A.5).

(b) The freedom in the available $\{\langle S_{abi_1\cdots i_{n+1}}, S_{abi_1\cdots i_n;0}\rangle\}$ is of the type (A.7), that is,

$$S_{abi_1\cdots i_{n+1}} \to S_{abi_1\cdots i_{n+1}} + \sigma_{abi_1\cdots i_{n+1}},$$
$$S_{abi_1\cdots i_n;0} \to S_{abi_1\cdots i_n;0} + \sigma_{abi_1\cdots i_n;0}$$

where the $\{\sigma_{abi_1\cdots i_{n+1}}, \sigma_{abi_1\cdots i_n;0}\}$ are arbitrary provided they satisfy (A.8), (A.9), and

$$\eta^{mn}\sigma_{abi_1\cdots i_{n-1}mn} = 0 \tag{A.19}$$

Also, equation (A.9) determines a one-to-one linear homogeneous transformation of the quantities $\{\sigma_{abi_1\cdots i_{n+1}}\}$ which satisfy (A.8) and (A.19) *onto* the 4-tuples $\{\langle \sigma_{\alpha\beta\gamma_1\cdots\gamma_{n+1}}, \sigma_{0\beta\gamma_1\cdots\gamma_{n+1}}, \sigma_{00\gamma_1\cdots\gamma_{n+1}}, \sigma_{abi_1\cdots i_n;0}\rangle\}$ of quantities that satisfy (A.10) and

$$\eta^{pq}\sigma_{abi_1\cdots i_{n-2}pq;0} = 0 \tag{A.20a}$$

$$\frac{n-1}{n+1}\sigma_{00\gamma_1\cdots\gamma_{n-1}0;0} - \sigma_{00\gamma_1\cdots\gamma_{n-1}\mu\mu} = 0 \tag{A.20b}$$

$$\frac{(n+2)(n-1)}{(n+1)^2}\sigma_{0\beta\gamma_1\cdots\gamma_{n-1}0;0} - \frac{n-1}{(n+1)^2}\sigma_{00\beta\gamma_1\cdots\gamma_{n-1};0} - \sigma_{0\beta\gamma_1\cdots\gamma_{n-1}\mu\mu} = 0 \tag{A.20c}$$

$$\frac{(n+2)(n-1)}{(n+1)n}\sigma_{\alpha\beta\gamma_1\cdots\gamma_{n-1}0;0} - \frac{2(n+2)(n-1)}{(n+1)^2 n}\sigma_{0(\alpha\beta)\gamma_1\cdots\gamma_{n-1};0} + \frac{2}{(n+1)n}\sigma_{00\alpha\beta\gamma_1\cdots\gamma_{n-1}} - \sigma_{\alpha\beta\gamma_1\cdots\gamma_{n-1}\mu\mu} = 0 \tag{A.20d}$$

The inverse transformation is determined by (A.11).

Remark. A special solution of (A.10) and (A.20) is when $\sigma_{\alpha\beta\gamma_1\cdots\gamma_{n+1}}$, $\sigma_{0\beta\gamma_1\cdots\gamma_{n+1}}$, $\sigma_{abi_1\cdots i_n;0}$ vanish and $\sigma_{00\gamma_1\cdots\gamma_{n+1}}$ satisfies $\sigma_{00\gamma_1\cdots\gamma_{n+1}} = \sigma_{00(\gamma_1\cdots\gamma_{n+1})}$, $\sigma_{00\gamma_1\cdots\gamma_{n-1}\mu\mu} = 0$.

The proof is a direct consequence of Theorem 2, Theorem 3, and Lemma 6. [Also, part (b) is a direct consequence of part (a).]

We know that generally $\sigma_{abi_1\cdots i_n;0}$ in (A.7) is arbitrary apart from the symmetries $\sigma_{abi_1\cdots i_n;0} = \sigma_{(ab)(i_1\cdots i_n);0}$, $\sigma_{a(bi_1\cdots i_n);0} = 0$, of (A.10). The vacuum condition introduces the further restrictions (A.20a). A question, important for our applications, is whether (A.20a) constitutes all the extra restrictions due to the vacuum condition over $\sigma_{abi_1\cdots i_n;0}$. In order to answer we have to check whether every $\sigma_{abi_1\cdots i_n;0}$ with the relevant symmetries included in (A.10) and with the "vacuum symmetries" (A.20a) can be associated with $\langle\sigma_{\alpha\beta\gamma_1\cdots\gamma_{n+1}}, \sigma_{0\beta\gamma_1\cdots\gamma_{n+1}}, \sigma_{00\gamma_1\cdots\gamma_{n+1}}\rangle$ with the symmetries of (A.10) such that the whole system (A.20) is satisfied. To this end we shall make use of the following three lemmas.

Lemma 7. For every n ($n = 1, 2, 3, \ldots$) and N ($N = 2, 3, 4, \ldots$), the equations

$$X_{\gamma_1\cdots\gamma_{n-1}\mu\mu} = A_{\gamma_1\cdots\gamma_{n-1}}$$
$$X_{\gamma_1\cdots\gamma_{n+1}} = X_{(\gamma_1\cdots\gamma_{n+1})}$$

in the unknowns $\{X_{\gamma_1\cdots\gamma_{n+1}}\}$ ($\gamma_i = 1, 2, \ldots, N$) are solvable *if and only if* $A_{\gamma_1\cdots\gamma_{n-1}} = A_{(\gamma_1\cdots\gamma_{n-1})}$.

Lemma 8. For every n ($n = 1, 2, \ldots$) and N ($N = 2, 3, \ldots$), the equations

$$X_{\alpha\gamma_1\cdots\gamma_{n-1}\mu\mu} = A_{\alpha\gamma_1\cdots\gamma_{n-1}}$$
$$X_{\alpha\gamma_1\cdots\gamma_{n+1}} = X_{\alpha(\gamma_1\cdots\gamma_{n+1})}$$
$$X_{(\alpha\gamma_1\cdots\gamma_{n+1})} = 0$$

in the unknowns $\{X_{\alpha\gamma_1\cdots\gamma_{n+1}}\}$, ($\alpha, \gamma_i = 1, \ldots, N$), are solvable *if and only if*

$$A_{\alpha\gamma_1\cdots\gamma_{n-1}} = A_{\alpha(\gamma_1\cdots\gamma_{n-1})}$$
$$4A_{\mu\mu\gamma_1\cdots\gamma_{n-2}} + (n-2)A_{(\gamma_1\cdots\gamma_{n-2})\mu\mu} = 0$$

Lemma 9. For every n ($n = 1, 2, 3, \ldots$) and N [$N = 3, 4, \ldots$(!)], the equations

$$X_{\alpha\beta\gamma_1\cdots\gamma_{n-1}\mu\mu} = A_{\alpha\beta\gamma_1\cdots\gamma_{n-1}}$$
$$X_{\alpha\beta\gamma_1\cdots\gamma_{n+1}} = X_{(\alpha\beta)(\gamma_1\cdots\gamma_{n+1})}$$
$$X_{\alpha(\beta\gamma_1\cdots\gamma_{n+1})} = 0$$

in the unknowns $\{X_{\alpha\beta\gamma_1\cdots\gamma_{n+1}}\}$ ($\alpha, \beta, \gamma_i = 1, \ldots, N$) are solvable *if and only if*

$$A_{\alpha\beta\gamma_1\cdots\gamma_{n-1}} = A_{(\alpha\beta)(\gamma_1\cdots\gamma_{n-1})} \tag{A.21a}$$
$$4A_{\alpha\mu\mu\gamma_1\cdots\gamma_{n-2}} + (n-2)A_{\alpha(\gamma_1\cdots\gamma_{n-2})\mu\mu} = 0 \tag{A.21b}$$

For $N = 2$ conditions (A.21) are necessary but not sufficient. The situation in the case $N = 2$ should not surprise us, since the dimension of "the $A_{\alpha\beta\gamma_1\cdots\gamma_{n-1}}$ space" exceeds the dimension of "the $X_{\alpha\beta\gamma_1\cdots\gamma_{n+1}}$ space" in this case ($N = 2$). The *detailed proofs* of Lemmas 7–9 are somewhat cumbersome. We offer some hints as to how to manage them according to our method and leave the details to the interested reader. The necessary conditions in the three lemmas follow immediately as consequences of some direct manipulations. The main parts of the proofs are to show sufficiency. To this end we make use, in all three cases, of a decomposition of symmetric quantities as follows. It is easy to show that (provided $N \geqslant 2$) every $S_{\gamma_1 \cdots \gamma_{n+1}} = S_{(\gamma_1 \cdots \gamma_{n+1})}$ can be represented as the sum

$$S_{\gamma_1 \cdots \gamma_{n+1}} = \sum_{r=0}^{M} S^{(r)}_{(\gamma_{2r+1} \cdots \gamma_{n+1}} \delta_{\gamma_1 \gamma_2} \cdots \delta_{\gamma_{2r-1}\gamma_{2r})}, \qquad M \equiv \left[\frac{n+1}{2}\right]$$

where $S^{(r)}_{(\gamma_{2r+1} \cdots \gamma_{n+1}}$ is totally symmetric and vanishes by one contraction of indices: $S^{(r)}_{\mu\mu\gamma_{2r+3} \cdots \gamma_{n+1}} = 0$. Also this decomposition is unique. Making use of this fact leads immediately to Lemma 7. We apply it also to the symmetric indices $\{\gamma_i\}$ of the X's and A's of Lemmas 8 and 9. Some difficulties arise only as to the possibility of determining the desired $X^{(0)}_{\alpha\gamma_1 \cdots \gamma_{n+1}}$ and $X^{(0)}_{\alpha\beta\gamma_1 \cdots \gamma_{n+1}}$, respectively. However, it is possible to construct these quantities as outer products of $\delta_{\mu\nu}$ and the available quantities appearing in the decomposition of the A's. Only the construction of $X^{(0)}_{\alpha\beta\gamma_1 \cdots \gamma_{n+1}}$ in the case $N = 2$ remains impossible as expected.

Now we prove the following theorem.

> *Theorem 5.* In vacuous space-time ($R_{ij} = 0$) of dimension 4 at least, the $\{\sigma_{abi_1 \cdots i_n;0}\}$ in (A.7) are arbitrary apart from the relevant symmetries included in (A.10) and (A.20a); that is, the vacuum condition introduces only the extra restrictions (A.20a) on the $\{\sigma_{abi_1 \cdots i_n;0}\}$.

Proof. Let $\sigma_{abi_1 \cdots i_n;0}$ with the relevant symmetries included in (A.10) and the symmetries (A.20a) be given. We have to show that there exist $\langle \sigma_{\alpha\beta\gamma_1 \cdots \gamma_{n+1}}, \sigma_{0\beta\gamma_1 \cdots \gamma_{n+1}}, \sigma_{00\gamma_1 \cdots \gamma_{n+1}} \rangle$ with the symmetries (A.10) such that (A.20) is satisfied. Indeed, Lemma 7 ensures that it is possible to find $\sigma_{00\gamma_1 \cdots \gamma_{n+1}}$ consistent with (A.10) and such that (A.20b) is satisfied. Lemma 8 ensures that it is possible to find $\sigma_{0\beta\gamma_1 \cdots \gamma_{n+1}}$ consistent with (A.10) and such that (A.20c) is satisfied. We may use this lemma since it is easy to show that our given $\sigma_{abi_1 \cdots i_n;0}$ ensures the sufficient condition of Lemma 8. Now we substitute the already-determined $\sigma_{00\gamma_1 \cdots \gamma_{n+1}}$ in (A.20d). Then, Lemma 9 ensures that it is possible to find $\sigma_{\alpha\beta\gamma_1 \cdots \gamma_{n+1}}$ consistent with (A.10) and such that (A.20d) is satisfied. We may use Lemma 9, since it is easy to show that our given $\sigma_{abi_1 \cdots i_n;0}$ and the already-fixed $\sigma_{00\gamma_1 \cdots \gamma_{n+1}}$ [which satisfies (A.20b)] ensure the sufficient condition of this lemma, (A.21). This completes the proof of Theorem 5.

Remark Concerning Theorem 5 in Riemannian Space of Dimension 2 or 3. A Riemannian space of dimension 2 or 3 does not constitute any real problem, since then the equation $R_{ab} = 0$ implies the vanishing of R_{abcd}. Therefore, all the $\{S_{abi_1 \cdots i_k}\}_{k=2}^{\infty}$ vanish (and, of course, the $\{\sigma_{abi_1 \cdots i_k}\}_{k=2}^{\infty}$ vanish too). Indeed the extra symmetries now imply the vanishing of the $\{\sigma_{abi_1 \cdots i_n;0}\}$; hence the theorem is still true.

REFERENCES

Enosh, M., and Kovetz, A. (1972). *Annals of Physics*, **69**, 279.
Enosh, M., and Kovetz, A. (1977). *International Journal of Theoretical Physics*. **16**, 895.
Penrose, R. (1960). *Annals of Physics*, **10**, 171.
Schouten, J. A. (1954). *Ricci Calculus*, p. 80. Springer-Verlag, Berlin.
Synge, J. L. (1960). *Relativity—The General Theory*, p. 123. North-Holland, Amsterdam.